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LARGE FACES IN 4-CRITICAL PLANAR GRAPHS WITH MINIMUM DEGREE 4

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We prove that the size of the largest face of a 4-critical planar graph with $\delta \geq 4$ is at most one half the number of its vertices. Let $f(n)$ denote the maximum of the sizes of largest faces of all such graphs with n vertices (n sufficiently large). We present an infinite family of graphs that shows $\lim_{n\to\infty} \frac{f(n)}{n} = \frac{1}{2}$.

A graph G is said to be k-critical if it has chromatic number k, but every proper subgraph of G has a $(k-1)$-coloring. In 1985, G. Koester [5] gave an example of a 4-critical 4-regular planar graph. It is the graph shown in Figure 1. This graph provided a counterexample to the old conjecture of Gallai [3] that every 4-critical planar graph has a vertex of degree 3. Koester's discovery sparked interest in the class $\mathcal{G}$ of 4-critical planar graphs with minimal degree $\delta \geq 4$. It is now known that there exist arbitrarily large 4-critical 4-regular planar graphs and that there exist graphs of order n in $\mathcal{G}$ for all sufficiently large n. See [2], [6], and [7]. It has also been shown by Koester [7] that every graph in $\mathcal{G}$ has a vertex of degree 4. For a different proof of this result see [1].

The aim of this paper is to investigate certain questions concerning the size of the largest face of graphs in $\mathcal{G}$. The odd wheel shows that there exist 4-critical planar graphs whose largest face contains all but one of the vertices. For $G \in \mathcal{G}$, however, the situation changes. Our main result is the following.

Theorem 1. *Let $G=(V,E)\in\mathcal{G}$. Then no face of G has more than $\frac{1}{2}|V|$ vertices.*

We shall also prove that Theorem 1 is, in a sense, best possible.

Theorem 2. *There exist absolute constants c_1 and c_2 such that for all $n \geq c_1$ there exists a graph G in $\mathcal{G}$ of order n whose largest face has size at least $\frac{1}{2}n - c_2$.*

In the proof of Theorem 1 we shall need the following simple lemma.

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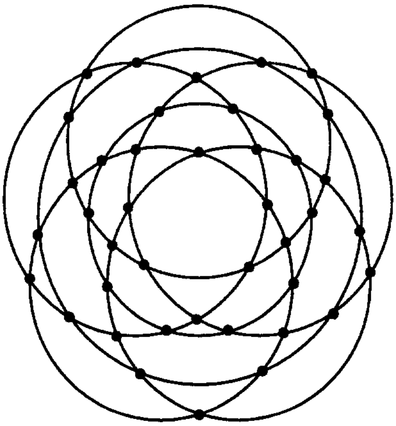

Fig. 1. Koester's graph

Lemma 3. *For $n \geq 3$, the graph H_n shown in Figure 2 is not a proper subgraph of any 4-critical graph.*

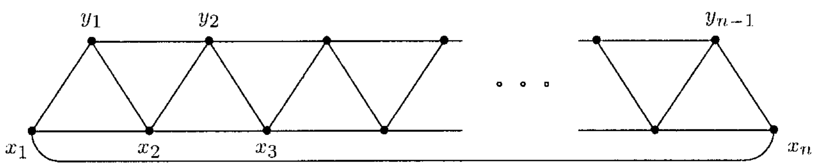

Fig. 2. The graph H_n

Proof. Suppose H_n is a proper subgraph of a 4-critical graph G for some $n \geq 3$. Then $n \not\equiv 1 \bmod 3$ since, as it is easy to check, H_n is itself 4-critical if $n \equiv 1 \bmod 3$. In any 3-coloring of $G - \{x_1 x_n\}$, x_1 and x_n are assigned the same color.

Such a 3-coloring induces a 3-coloring of $H_n - \{x_1 x_n\}$. However, it is straightforward to verify that in any 3-coloring of $H_n - \{x_1 x_n\}$, x_1 and x_n must be assigned different colors when $n \equiv 0$, $2 \bmod 3$. ∎

Proof of Theorem 1. Consider a plane drawing of G in which the infinite face is a largest face. We denote the infinite face and its set of vertices by F. There should be no danger of confusion. Let $S = V \setminus F$. For each $v \in V$ and $X \subseteq V$, let $d_X(v)$ denote the number of edges from v to X.

Suppose that $d_S(v) \leq 1$ for some vertex v of F. Then, since $\delta = 4$, there is a vertex u of F such that $uv \in E$, but uv is not an edge of F. There are then two proper subgraphs G_1 and G_2 of G such that $V(G_1) \cap V(G_2) = \{u, v\}$, $E(G_1) \cap E(G_2) = \{uv\}$ and $G_1 \cup G_2 = G$. There is a 3-coloring of G_1 in which u is colored red and v is colored blue and a 3-coloring of G_2 with the same property. This yields a 3-coloring of G. If follows that $d_S(v) \geq 2$ for all vertices v of F.

If for all vertices v of S we have $d_F(v) \leq 2$ then

$$2|F| \leq \sum_{v\in F} d_S(v) = \sum_{v\in S} d_F(v) \leq 2|S|$$

so that $|F| \leq |S|$ and the theorem holds. We may therefore suppose that $d_F(v) \leq 2$ does not hold for all $v \in S$.

Let $w \in S$ be a fixed vertex satisfying $d_F(w) = m \geq 3$ and let the neighbors of w on F be $w_1, w_2, \ldots, w_m$, listed in some counterclockwise order (the choice of which vertex to label w_1 is arbitrary).

Consider any $x \in S$, $x \neq w$, such that $d_F(x) = k \geq 2$. The neighbors of x on F lie between w_i and w_{i+1} for some $i \in \{1,2,\ldots,m\}$ (here $w_{m+1} = w_1$). Label the neighbors of x on F $x_1, x_2, \ldots, x_k$ in the counterclockwise order that has x_1 as the first vertex encountered after w_i when traversing along F in the counterclockwise direction. It may happen that $x_i = w_i$ or $x_k = w_{i+1}$.

Let G^x denote the subgraph of G induced by the set of vertices in the interior of or on the boundary of the region whose boundary consists of the edges xx_1 and xx_k and the part of the boundary of F joining x_1 to x_k in the counterclockwise sense. Let $F^x = (V(G^x) \cap F) \setminus \{x_1, x_k\}$ and let $S^x = V(G^x) \setminus (F^x \cup \{x, x_1, x_k\})$. See Figure 3.

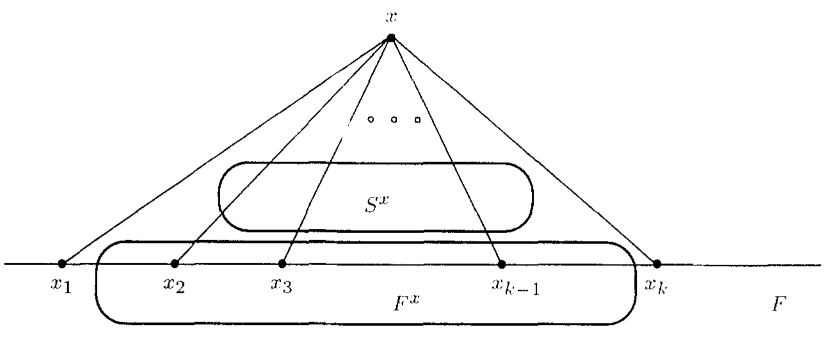

Fig. 3. The subgraph G^x

We define the *depth* of x as follows: $depth(x) = 0$ if there is no vertex z of S^x such that $d_F(z) \geq 2$ and for $l \geq 0$, $depth(x) = l + 1$ where $l = \max\{depth(z) : z \in S^x, d_F(z) \geq 2\}$. Note that if $d_F(x) < 2$, then $depth(x)$ is left undefined.

Define

$$g^x = \begin{cases} 0 & \text{if } d_{S^x}(x_1) = 0 \text{ and } d_{S^x}(x_k) = 0, \\ 1 & \text{if } d_{S^x}(x_1) \geq 1 \text{ or } d_{S^x}(x_k) \geq 1, \text{ but not both}, \\ 2 & \text{if } d_{S^x}(x_1) \geq 1 \text{ and } d_{S^x}(x_k) \geq 1. \end{cases}$$

The bulk of the remainder of the argument involves establishing

(1) $$|F^x| = |S^x| \text{ implies } F^x = \emptyset = S^x$$

and

(2) $$|F^x| \leq |S^x| - g^x - 1 \text{ whenever } S^x \neq \emptyset.$$

We prove (1) and (2) by induction on $depth(x)$. Let $depth(x) = 0$. By definition, $d_F(z) \leq 1$ for all $z \in S^x$. For $i = 1, 2, \ldots, k-1$, let S_i^x be the set of those vertices of S^x lying in the interior of the region whose boundary consists of the edges xx_i and xx_{i+1} and the part of the boundary of F joining x_i to x_{i+1} in the counterclockwise sense. Let F_i^x consist of those vertices of F^x between x_i and x_{i+1}, not including x_i and x_{i+1}. See Figure 4.

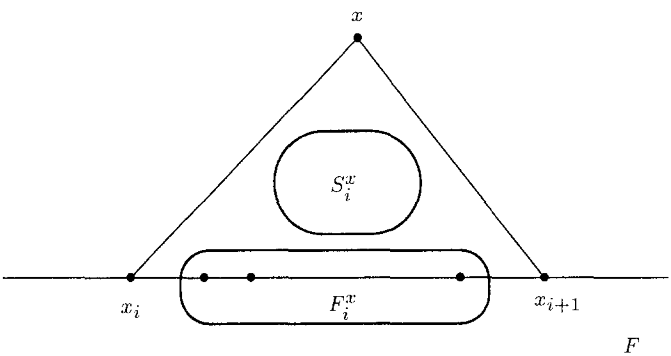

Fig. 4. The sets S_i^x and F_i^x

For $i = 1, 2, \ldots, k-1$, we have

$$2|F_i^x| + d_{S_i^x}(x_i) + d_{S_i^x}(x_{i+1}) \leq \sum_{v \in F_i^x} d_S(v) + d_{S_i^x}(x_i) + d_{S_i^x}(x_{i+1})$$

$$= \text{number of edges from } F_i^x \cup \{x_i, x_{i+1}\} \text{ to } S_i^x = \sum_{v \in S_i^x} d_F(v) \leq |S_i^x|.$$

This gives

(3) $$2|F_i^x| \leq |S_i^x| - d_{S_i^x}(x_i) - d_{S_i^x}(x_{i+1}).$$

Partition $\{1, 2, \ldots, k-1\}$ into three sets I_0, I_1, I_2 as follows:

$$I_0 = \{i : d_{S_i^x}(x_i) = d_{S_i^x}(x_{i+1}) = 0\}$$
$$I_1 = \{i : d_{S_i^x}(x_i) \geq 1, d_{S_i^x}(x_{i+1}) = 0\} \cup \{i : d_{S_i^x}(x_i) = 0, d_{S_i^x}(x_{i+1}) \geq 1\}$$
$$I_2 = \{i : d_{S_i^x}(x_i) \geq 1, d_{S_i^x}(x_{i+1}) \geq 1\}.$$

By (3),

(4) $$|F_i^x| \leq |S_i^x| \text{ for all } i \in I_0.$$

If $i \in I_1 \cup I_2$, then $S_i^x \neq \emptyset$ and since $d_F(s) \leq 1$ for all $s \in S_i^x$ and $\delta = 4$, $|S_i^x| \geq 3$. Hence (3) gives

$$|F_i^x| \leq |S_i^x| - 2 \text{ for all } i \in I_1 \tag{5}$$

and

$$|F_i^x| \leq |S_i^x| - 3 \text{ for all } i \in I_2. \tag{6}$$

It now follows from inequalities (4), (5), and (6) that

$$\begin{aligned} |F^x| &= k - 2 + \sum_{i=1}^{k-1} |F_i^x| \\ &= k - 2 + \sum_{i \in I_0} |F_i^x| + \sum_{i \in I_1} |F_i^x| + \sum_{i \in I_2} |F_i^x| \\ &\leq k - 2 + \sum_{i \in I_0} |S_i^x| + \sum_{i \in I_1} (|S_i^x| - 2) + \sum_{i \in I_2} (|S_i^x| - 3) \\ &= k - 2 + \sum_{i=1}^{k-1} |S_i^x| - 2|I_1| - 3|I_2|. \end{aligned} \tag{7}$$

Moreover since $\delta = 4$,

$$\begin{aligned} 0|I_0| + 1|I_1| + 2|I_2| &= \Big|\{i : d_{S_i^x}(x_i) \geq 1\}\Big| + \Big|\{i : d_{S_i^x}(x_{i+1}) \geq 1\}\Big| \\ &\geq k - 2 + g^x. \end{aligned} \tag{8}$$

Combining (7) with (8) gives

$$|F^x| \leq |S^x| - g^x - |I_1| - |I_2|. \tag{9}$$

If $|F^x| = |S^x|$, then by (9) $g^x = 0$ and $I_1 = \emptyset = I_2$. From (8) we have $k = 2$ and hence (3) implies $S^x = \emptyset$. Thus (1) is established in the case $depth(x) = 0$.

Consider now the case when $S^x \neq \emptyset$. If $g^x = 0$, then by the argument in the previous paragraph $|F^x| \neq |S^x|$, and if $g^x \geq 1$, then $|I_1| + |I_2| \geq 1$. Thus (9) establishes (2) in the case $depth(x) = 0$.

Suppose now that $l > 0$ and that (1) and (2) have been established for all vertices x of $S \backslash \{w\}$ satisfying $d_F(x) \geq 2$ and $depth(x) < l$. Let $x \in S \backslash \{w\}$, $d_F(x) \geq 2$ and $depth(x) = l$. (If there is no such x, (1) and (2) are established). We adopt the notation used in the case $depth(x) = 0$. Let Z^x be the set of all vertices z of S^x such that $d_F(z) \geq 2$ and such that for all z' of S^x satisfying $z' \neq z$ and $d_F(z') \geq 2$, G^z is not a subgraph of $G^{z'}$. Let $Z^x = \{z_1, z_2, \ldots, z_h\}$. Let a_j be the "left most" vertex and b_j the "right most" vertex of G^{z_j} on F and suppose that the elements of Z^x are labelled so that a_{j+1} does not precede b_j in the counterclockwise order

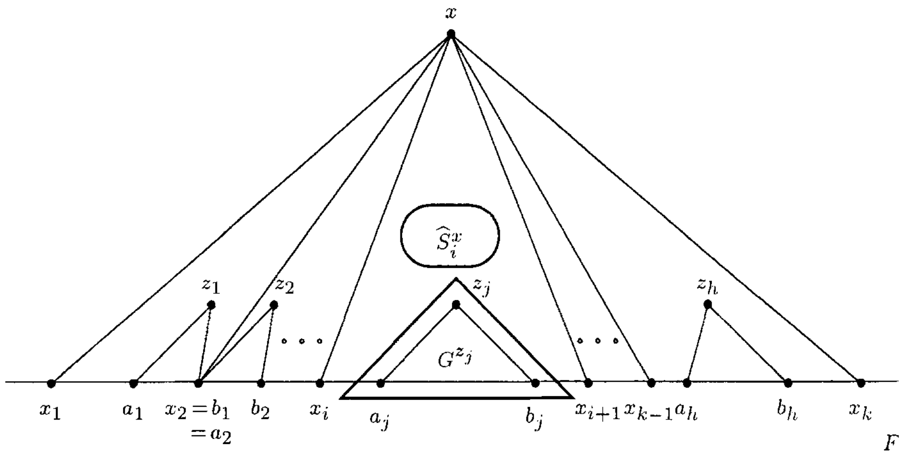

Fig. 5. The labelling of vertices of G^{z_j}

on F. It may happen that $a_{j+1}=b_j$. It may also happen that $x_i=a_j$ or b_j for some i, j. See Figure 5.

For $i=1,\,2,\,\ldots,\,k-1$, let

$$\widehat{S}_i^x = S_i^x \setminus \left(\bigcup_{j=1}^{h} V(G^{z_j})\right)$$

$$\widehat{F}_i^x = F_i^x \setminus \left(\bigcup_{j=1}^{h} V(G^{z_j})\right)$$

$$J_i = \{j : 1 \le j \le h, V(G^{z_j}) \cap S^x \subseteq S_i^x\}$$

and

$$L_i = \left\{j \in J_i : V(G^{z_j}) \cap S^x = \{z_j\}\right\}.$$

Observe that for $j \in L_i$, $V(G^{z_j}) \cap F = \{a_j, b_j\}$.

By the induction hypothesis, since $depth(z_j) < l$, for each $j \in J_i \setminus L_i$,

$$\left|(V(G^{z_j}) \cap F) \setminus \{a_j,b_j\}\right| \le \left|(V(G^{z_j}) \cap S_i^x) \setminus \{z_j\}\right| - g^{z_j} - 1.$$

We therefore get

$$\sum_{j\in J_i} |V(G^{z_j}) \cap F| \le \sum_{j\in J_i} |V(G^{z_j}) \cap S_i^x| - \sum_{j\in J_i} g^{z_j} + |L_i|. \tag{10}$$

For $i=1,\,2,\,\ldots,\,k-1$, let $p_i = d_{\widehat{S}_i^x}(x_i) + d_{\widehat{S}_i^x}(x_{i+1})$, $q_i = \left|\{j \in J_i : a_j = b_{j-1}\}\right|$ and $r_i = \left|\{x_i, x_{i+1}\} \cap \{a_j, b_j : j \in J_i\}\right|$. Observe that $r_i = 0$, 1 or 2 and that $|\widehat{S}_i^x| \ge p_i$.

Since $d_F(s) \le 1$ for all $s \in \widehat{S}_i^x$, the number of edges from F to $\widehat{S}_i^x$ is at most $|\widehat{S}_i^x|$. There are at least $2|\widehat{F}_i^x|$ edges from $\widehat{F}_i^x$ to $\widehat{S}_i^x$. There are p_i edges from $\{x_i, x_{i+1}\}$ to $\widehat{S}_i^x$. For each $j \in J_i$ there are at least $2 - g^{z_j}$ edges from $\{a_j, b_j\}$ to vertices not in $G^{z_j} \cup F$ and thus at least $\sum_{j \in J_i}(2 - g^{z_j}) - 2q_i - r_i$ of the edges of this sort from F to $\widehat{S}_i^x$. It follows that

$$2|\widehat{F}_i^x| + p_i + \sum_{j \in J_i}(2 - g^{z_j}) - 2q_i - r_i \le |\widehat{S}_i^x|.$$

This implies

$$|\widehat{F}_i^x| \le \frac{1}{2}|\widehat{S}_i^x| - |J_i| + \frac{1}{2}\sum_{j \in J_i} g^{z_j} + \frac{r_i}{2} - \frac{p_i}{2} + q_i. \tag{11}$$

Using (10) and (11) we have

$$\begin{aligned}
|F_i^x| &= |\widehat{F}_i^x| + \left| \bigcup_{j \in J_i} (V(G^{z_j}) \cap F) \right| - r_i \\
&= |\widehat{F}_i^x| + \sum_{j \in J_i} |V(G^{z_j}) \cap F| - q_i - r_i \\
&\le |\widehat{F}_i^x| + \sum_{j \in J_i} |V(G^{z_j}) \cap S_i^x| - \sum_{j \in J_i} g^{z_j} + |L_i| - q_i - r_i \\
&\le \frac{1}{2}|\widehat{S}_i^x| + \sum_{j \in J_i} |V(G^{z_j}) \cap S_i^x| - \frac{1}{2}\sum_{j \in J_i} g^{z_j} - |J_i \setminus L_i| - \frac{r_i}{2} - \frac{p_i}{2} \\
&= |S_i^x| - \frac{1}{2}\left\{ |\widehat{S}_i^x| + \sum_{j \in J_i \setminus L_i} (g^{z_j} + 2) + r_i + p_i \right\}. \qquad (12)
\end{aligned}$$

At the last step, in rewriting the sum, we use the fact that $g^{z_j} = 0$ when $j \in L_i$. It follows from (12) that

$$|F_i^x| \le |S_i^x| \text{ for all } i \in I_0.$$

That is, (4) holds when $depth(x) = l$.

Next we show that (5) holds when $depth(x) = l$.

If $i \in I_1$, then $r_i + p_i \ge 1$ and hence from (12) we have $|F_i^x| \le |S_i^x| - 1$. Suppose for some $i \in I_1$, $|F_i^x| = |S_i^x| - 1$. From (12) it then follows that

$$|\widehat{S}_i^x| + \sum_{j \in J_i \setminus L_i} (g^{z_j} + 2) + r_i + p_i \le 2.$$

This is possible only if the sum on the left is empty; that is, $J_i = L_i$. We then get

$$|\widehat{S}_i^x| + r_i + p_i \leq 2.$$

If $p_i > 0$, then $|\widehat{S}_i^x| > 0$ and thus $p_i = |\widehat{S}_i^x| = 1$ and $r_i = 0$. Thus $F_i^x = \emptyset$ and $S_i^x = \widehat{S}_i^x = \{s\}$ for some $s \in S$. But this implies $d(s) \leq 2$. Thus $p_i = 0$. We must then have $r_i = 1$ (since $r_i = 2$ would imply $i \in I_2$) and $|\widehat{S}_i^x| \leq 1$.

If $|\widehat{S}_i^x| = 0$, then since $J_i = L_i$, $d_S(v) = 2$ for all $v \in F_i^x$. Moreover we have $S_i^x = \{z_j : j \in L_i\}$ and thus $d_F(s) = 2$ for all $s \in S_i^x$. Counting the edges between S_i^x and F_i^x gives the contradiction $2|S_i^x| - r_i = 2|F_i^x|$. Therefore $\widehat{S}_i^x = \{s\}$ for some $s \in S$.

It follows that there is a chain from one of x_i or x_{i+1} to s containing all of the vertices of $S_i^x \cup F_i^x$ and whose interior vertices are alternately from S_i^x and F_i^x. Without loss of generality, suppose the chain starts at x_i. Relabel the vertices of S_i^x, $s_1, s_2, \ldots, s_t = s$, as shown in Figure 6.

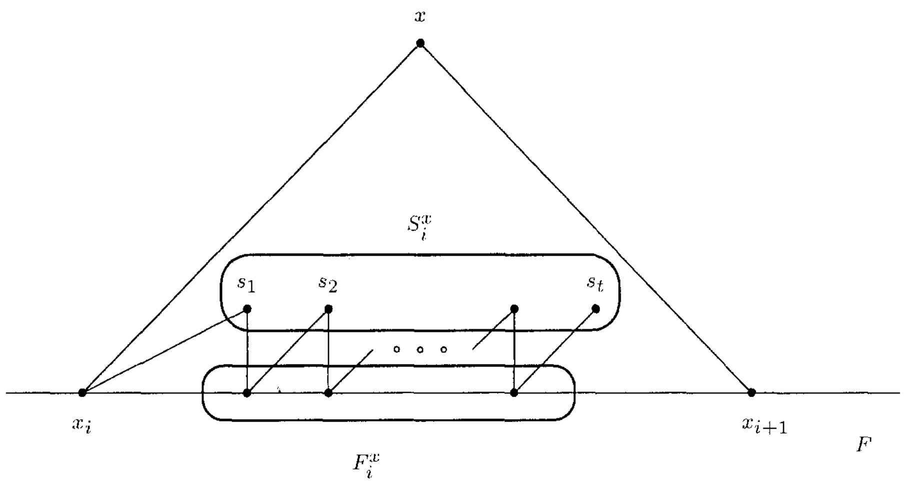

Fig. 6. The case $i \in I_1$ and $|F_i^x| = |S_i^x| - 1$

Suppose that for a pair of non-consecutive integers a and b in $\{1, 2, \ldots, t\}$, $a < b$, $s_a s_b$ is an edge of G and choose such a pair with $b - a$ minimal. The condition $\delta = 4$ then ensures that $s_a s_{a+1}, s_{a+1} s_{a+2}, \ldots, s_{b-1} s_b$ are edges of G. This implies that H_{b-a+1} is a proper subgraph of G, contrary to the lemma. It follows that no such pair a, b exists. Equivalently, if $s_a s_b \in E$ for some $a, b \in \{1, 2, \ldots, t\}$, $a < b$, then $b = a + 1$.

Since s is not adjacent to x_{i+1} and by the previous paragraph s is adjacent to at most one vertex from S_i^x, $d(s) \leq 3$. This contradiction establishes (5) in the case $depth(x) = l$.

We now show (6) holds when $depth(x)=l$. If $i\in I_2$, then $r_i+p_i\geq 2$ and thus (12) implies $|F_i^x|\leq|S_i^x|-1$. Suppose that for some $i\in I_2$, $|F_i^x|=|S_i^x|-1$. Then (12) implies that $r_i+p_i=2$, $\widehat{S}_i^x=\emptyset$ and $L_i=J_i$. Thus $p_i=0$ and G^x must contain the subgraph shown in Figure 7.

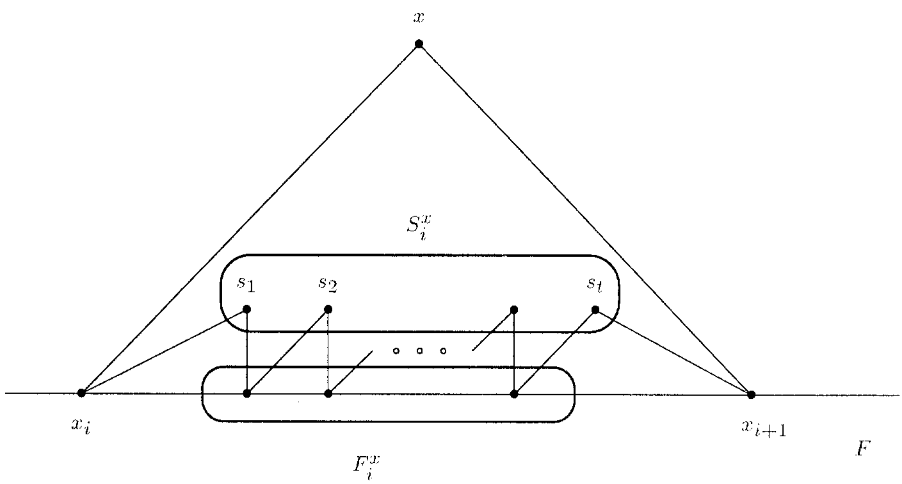

Fig. 7. The case $i\in I_2$ and $|F_i^x|=|S_i^x|-1$

As in the case when $i\in I_1$, if $s_as_b\in E$ for some $a,b\in\{1,2,\ldots,t\}$, $a<b$, then $b=a+1$. The condition $\delta=4$ implies that xs_1, s_1s_2, xs_t and $s_{t-1}s_t$ are edges of G. Let j be the least integer such that $j>1$ and $xs_j\in E$. It is straightforward to check that if $j\equiv 0$ mod 3 there is no 3-coloring of G^x. In attempting to effect such a 3-coloring one must assign different colors to x, x_i and s_1 and the structure of G^x is such that the colors assigned to s_2, s_3, $\ldots$, s_{j-1} and their neighbors on F_i^x are forced and three of the neighbors of s_j have different colors. One also finds that if $j\equiv 1$ mod 3, there is no 3-coloring of $G-\{xs_1\}$ in which x and s_1 have the same color, and if $j\equiv 2$ mod 3, there is no 3-coloring of $G-\{xx_i\}$ in which x and x_i have the same color. This contradiction thus establishes $|F_i^x|\leq|S_i^x|-2$ for all $i\in I_2$. The argument used in this paragraph will be used, almost verbatim, at a later stage of the proof. We refer to it as (A), so as to avoid having to repeat it.

Suppose that for some $i\in I_2$, $|F_i^x|=|S_i^x|-2$. Then, by (12),

$$|\widehat{S}_i^x|+\sum_{j\in J_i\setminus L_i}(g^{z_j}+2)+r_i+p_i\leq 4.$$

Since $r_i+p_i\geq 2$, we must have $J_i=L_i$. Otherwise, the sum on the left is nonempty and thus at least 3, since $g^{z_j}\neq 0$ for $j\notin L_i$. It follows that

$$|\widehat{S}_i^x|+r_i+p_i\leq 4.$$

If $r_i = 0$, then $p_i \geq 2$ and hence $|\widehat{S}_i^x| \geq p_i \geq 2$. It follows that $|\widehat{S}_i^x| = p_i = 2$. If $L_i \neq \emptyset$, then since $r_i = 0$ at least 2 vertices from F_i^x have neighbors in $\widehat{S}_i^x$ and since $d_F(s) \leq 1$ for all $s \in \widehat{S}_i^x$, we must have $|\widehat{S}_i^x| > 2$, a contradiction. Thus $L_i = \emptyset$. But then $d(s) \leq 3$ for every $s \in \widehat{S}_i^x$, another contradiction. Hence $r_i \geq 1$.

Suppose $r_i = 1$. Then $p_i + |\widehat{S}_i^x| \leq 3$ and $|\widehat{S}_i^x| \geq p_i \geq 1$, so that $p_i = 1$ and $|\widehat{S}_i^x|$ is either 1 or 2. If $|\widehat{S}_i^x| = 1$, then since $p_i = 1$ and $J_i = L_i$, $\{z_j : j \in L_i\}$ is the set of neighbors in S of the vertices in F_i^x. Hence $d_{S_i^x}(v) = 2$ for all $v \in F_i^x$ and $d_F(s) = 2$ for all $s \in S_i^x \setminus \widehat{S}_i^x$. Counting the edges between S_i^x and F_i^x gives the contradiction $2(|S_i^x| - 1) - r_i = 2|F_i^x|$. Therefore $|\widehat{S}_i^x| = 2$.

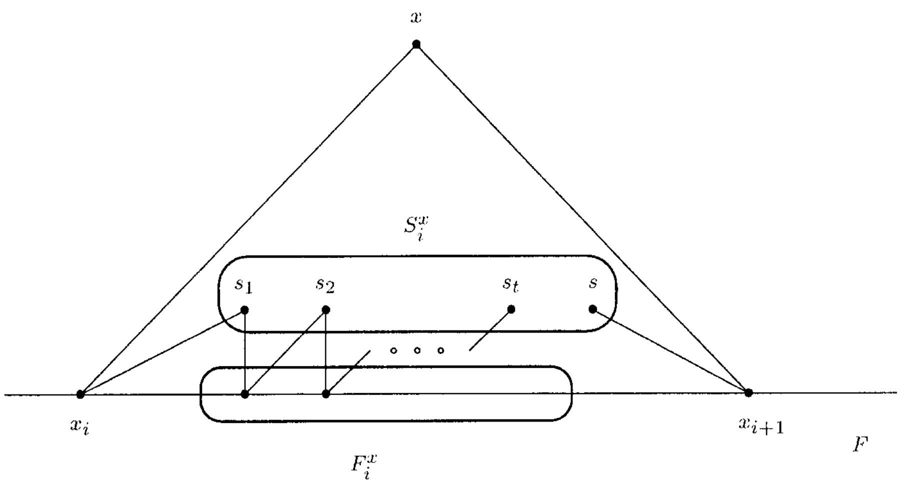

Fig. 8. The case $i \in I_2$, $|F_i^x| = |S_i^x| - 2$ and $r_i = 1$

Thus G^x must contain the subgraph shown in Figure 8. Here $x_1 s_1$ is the edge counted by r_i, s_t and s are the vertices of $\widehat{S}_i^x$, $x_{i+1}s$ is the edge counted by p_i and $s_1, s_2, \ldots, s_{t-1}$ are the vertices in Z^x.

If xs_t is an edge of G, then $d(s) \leq 3$. Hence s_t must be adjacent to at least two vertices from $s_1, s_2, \ldots, s_{t-1}$. This implies that G must contain one of the graphs described in the lemma, a contradiction. Thus $r_i = 2$.

This gives $p_i + |\widehat{S}_i^x| \leq 2$ and since $|\widehat{S}_i^x| \geq p_i$, we are left with four possibilities: $p_i = 0$ and $|\widehat{S}_i^x|$ is either 0,1 or 2, or $p_i = 1$ and $|\widehat{S}_i^x| = 1$.

If $p_i = |\widehat{S}_i^x| = 0$, then arguing as before we count the edges between S_i^x and F_i^x to obtain the contradiction $2|S_i^x| - r_i = 2|F_i^x|$.

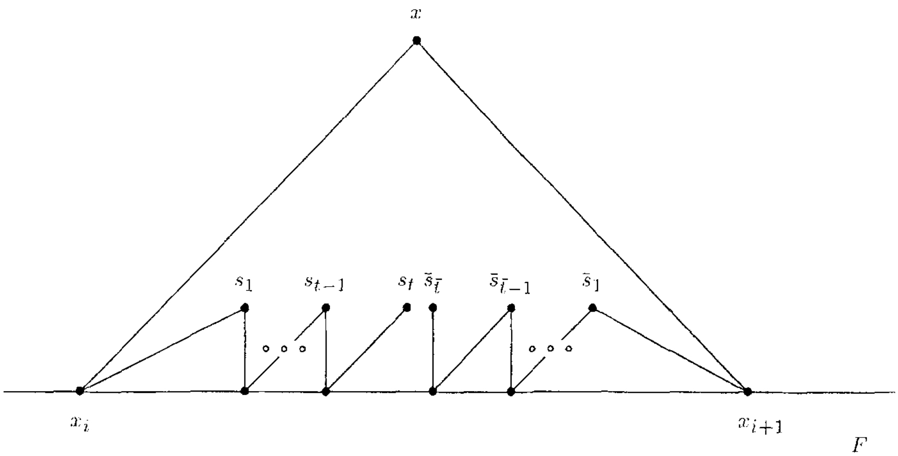

Fig. 9. The case $i \in I_2$, $|F_i^x| = |S_i^x| - 2$, $r_i = 2$ and $p_i = 0$

Suppose $p_i = 0$ and $|\widehat{S}_i^x| = 2$. G^x must then contain the subgraph shown in Figure 9. Here $x_i s_1$ and $x_{i+1}\bar{s}_1$ are the edges counted by $r_i = 2$, s_t and $\bar{s}_{\bar{t}}$ are the vertices of $\widehat{S}_i^x$ and $s_1, s_2, \ldots, s_{t-1}, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_{\bar{t}-1}$ are the vertices of Z^x. The vertex s_t cannot have more than one neighbor among $s_1, s_2, \ldots, s_{t-1}$ and if it has exactly one such neighbor, it must be s_{t-1}. Otherwise, G would contain one of the forbidden subgraphs described in the lemma. Also, since G is planar and $\delta = 4$, s_t cannot be adjacent to any of $\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_{\bar{t}-1}$. Thus s_t is adjacent to x, s_{t-1} and $\bar{s}_{\bar{t}}$. The condition $\delta = 4$ then implies that xs_1, $s_1 s_2$ are edges of G. It is now easy to see that argument (A) may be used to get a contradiction.

Therefore we must have $p_i \leq 1$ and $|\widehat{S}_i^x| = 1$. These two cases are similar and so we only present the argument for $p_i = |\widehat{S}_i^x| = 1$. In this case, G^x must contain the subgraph shown in Figure 10. Here $x_i s_1$ and $x_{i+1}s_{t-1}$ are the edges counted by r_i, $x_{i+1}s_t$ is the edge counted by p_i and s_t is the only vertex of $\widehat{S}_i^x$. The vertex s_t must have at least two neighbors among $s_1, s_2, \ldots, s_{t-1}$ and one sees that G must contain one of the subgraphs described in the lemma. Thus (6) is established when $depth(x) = l$.

Continuing as in the proof of the case when $depth(x) = 0$, it follows that (9) is established when $depth(x) = l$. If $|F^x| = |S^x|$, then as in the case when $depth(x) = 0$, $k = 2$ and $S^x = S_1^x$, $\widehat{S}_1^x = \emptyset$, $r_i = 0 = p_i$ and $L_i = J_i$. But $r_i = 0$ implies that if $L_i \neq \emptyset$, then at least 2 vertices from F_1^x have neighbors in $\widehat{S}_1^x$, a contradiction. Thus $L_i = \emptyset$ and (1) is established when $depth(x) = l$. Finishing off as in the case $depth(x) = 0$, (2) is established when $depth(x) = l$.

The proof of the theorem may now be easily completed. For $i = 1, 2, \ldots, m$ let G_i^w be the subgraph of G induced by the vertices in the interior of or on the

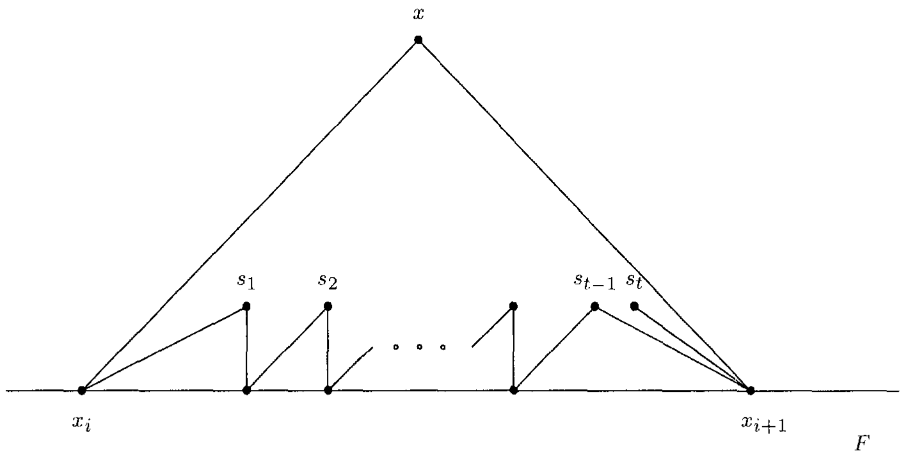

Fig. 10. The case $i \in I_2$, $|F_i^x| = |S_i^x| - 2$, $r_i = 2$ and $p_i = 1$

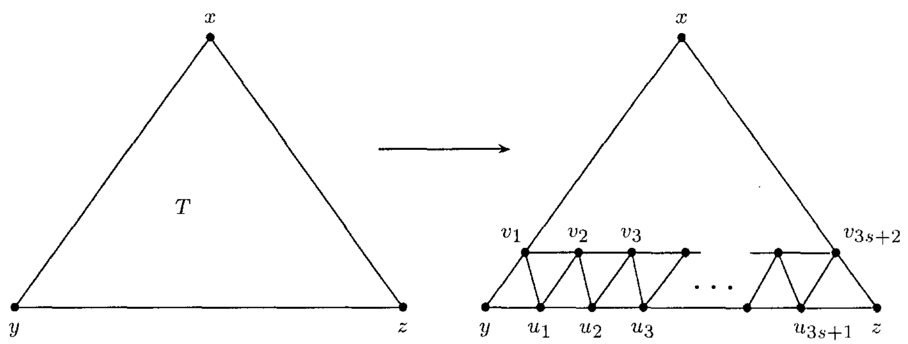

Fig. 11. The transformation of T

boundary of the region whose boundary consists of the edges ww_i, ww_{i+1} and the part of the boundary of F containing the remaining neighbors of w. Define g_i^w analogously. It is understood that $w_{m+1} = w_1$. Then (1) and (2) hold for G_i^w. Thus

$$|V(G_i^w) \cap F| \le |V(G_i^w) \cap S| + 1 - g_i^w.$$

Since $\delta = 4$, $\sum_{i=1}^{m} g_i^w \ge m$ and hence

$$m + (m-1)|F| = \sum_{i=1}^{m} |V(G_i^w) \cap F| \le m + \sum_{i=1}^{m} |V(G_i^w) \cap S| - \sum_{i=1}^{m} g_i^w \le (m-1)|S| + 1.$$

This gives $|F| \le |S| - 1$ and Theorem 1 is proved. ∎

Proof of Theorem 2. Let $G \in \mathcal{G}$ be a graph of order h and suppose that G has a triangular face T. There are such graphs in $\mathcal{G}$ for any $h \geq 81$ (see [2] or [7]). Let T have vertices x, y, z. Delete the edges xy, xz, yz and add new vertices v_1, v_2, $\ldots$, v_{3s+2}, u_1, u_2, $\ldots$, u_{3s+1} and new edges xv_1, xv_{3s+2}, yv_1, yu_1, zv_{3s+2}, zu_{3s+1}; v_iv_{i+1}, v_iu_i, u_iv_{i+1}, $i=1, 2, \ldots, 3s+1$, u_iu_{i+1}, $i=1, 2, \ldots, 3s$. See Figure 11.

Denote the resulting graph by G_s. It is straightforward to check that $G_s \in \mathcal{G}$ for all $s \geq 1$. G_s has order $n=h+6s+3$ and the largest face of G_s has size at least $3s+3=\frac{n}{2}-\frac{h-3}{2}$. ∎

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